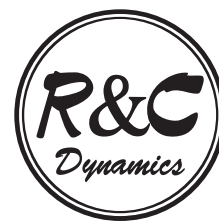


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ON THE INTEGRATION THEORY OF EQUATIONS OF NONHOLONOMIC MECHANICS*

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The paper deals with the problem of integration of equations of motion in nonholonomic systems. By means of well-known theory of the differential equations with an invariant measure the new integrable systems are discovered. Among them there are the generalization of Chaplygin's problem of rolling nonsymmetric ball in the plane and the Suslov problem of rotation of rigid body with a fixed point. The structure of dynamics of systems on the invariant manifold in the integrable problems is shown. Some new ideas in the theory of integration of the equations in nonholonomic mechanics are suggested. The first of them consists in using known integrals as the constraints. The second is the use of resolvable groups of symmetries in nonholonomic systems. The existence conditions of invariant measure with analytical density for the differential equations of nonholonomic mechanics is given.

1. Introduction

The integration theory of equations of motion for mechanical systems with nonholonomic constraints isn't so complete as in the case of systems with holonomic constraints. This fact has many reasons. First, the equations of nonholonomic mechanics have a more complex structure than the Lagrange equations, which describe the dynamics of systems with integrable constraints. For example, a nonholonomic system can't be characterized by the only function of its state and time (cf. [1], ch. XXIV). Second, the equations of nonholonomic mechanics have no invariant measure in the general case (a simple example is given in section 5). The point is that nonholonomic constraints may be realized by action of complementary forces of viscous anisotropic friction with a large viscosity coefficient ([3]). The absence of an invariant measure is a characteristic property of systems with friction. In limit, the anisotropic friction is compatible with the conservation of total energy. But asymptotically stable equilibriums or limit cycles may arise on the manifolds of energy levels (cf. [4]), and this is the reason for nonexistence of additional "regular" integrals of motion.

The most popular method to integrate the equations of nonholonomic dynamics is based on the use of the available first integrals or the "conservation laws": if a Lie group that acts on a position space preserves the Lagrangian and if vector fields that generate this group are the fields of possible velocities then the equations of motion have the first "vector" integral or the generalized integral of kinetic moment [5,6]. A number of problems of nonholonomic dynamics was solved by this method, among them, we note especially Chaplygin's problem on an asymmetrical ball rolling over a horizontal plane [5].

Attempts to generalize the Hamilton–Jacobi method to the systems with nonholonomic constraints were non-effective as well as attempts to present the equations of nonholonomic dynamics in the form of Hamiltonian canonical equations. It turned out that with the help of the Hamilton–Jacobi

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generalized method it is possible to find at most only some special solutions of the motion equations. This paper contains the detailed analysis of these questions.

Another general approach to the integration of nonholonomic equations is based on the theory of Chaplygin's reducing factor ([5]): one try to obtain a change of time (different along different trajectories), such that the equations of motion are presented as Lagrange or Hamilton equations. Though such change exists in exceptional cases only, it allows to solve a number of new problems of nonholonomic dynamics (cf. [5]). Let us note that the equations of motion sometimes may be reduced to the Hamiltonian form by other reasoning (see section 5).

The list of exactly solvable problems of nonholonomic dynamics isn't long: almost complete information may be found in the books [1,5,8]. In this work we present some new integrable problems, consider the characteristic features of behavior of nonholonomic systems' trajectories in the phase space, and propose some general theoretical reasonings on methods of integrating the equations of nonholonomic dynamics.

2. Differential equations with an integrable measure

Let us consider a differential equation

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n \quad (2.1)$$

and let g^t be its phase flow. Suppose (2.1) has an integral invariant with some smooth density $M(x)$, i.e. for any measurable domain $D \subset \mathbb{R}^n$ the following equation holds for all t

$$\int_{g^t(D)} M(x) dx = \int_D M(x) dx. \quad (2.2)$$

Let us recall the well-known Liouville assertion: a smooth function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is a density of an invariant

$$\int M(x) dx$$

if and only if $\operatorname{div}(Mf) \equiv 0$. If $M(x) > 0$ for all x then (2.2) defines a measure in \mathbb{R}^n invariant with respect to the action of f . The existence of an invariant measure simplifies the integration of a differential equation; for example, in the case of $n = 2$ the equation is always integrable in quadratures. According to Euler, M is also referred to as an integrating factor.

Theorem 1. *Suppose system (2.1) with invariant measure (2.2) has $n - 2$ first integrals F_1, \dots, F_{n-2} . Suppose the functions F_1, \dots, F_{n-2} are independent on the invariant set $E_c = \{x \in \mathbb{R}^n: F_s(x) = c_s, 1 \leq s \leq n - 2\}$.*

Then

1) *the solutions of (2.1) that belong to E_c may be found by quadratures. If L_c is a connected compact component of the level set E_c and $f \neq 0$ on L_c then*

2) *L_c is a smooth surface diffeomorphic to a two-dimensional torus,*

3) *it is possible to choose angle variables $x, y \bmod 2\pi$ on L_c so that, after the change of variables, system (2.1) on L_c would have the following form*

$$\dot{x} = \frac{\lambda}{\Phi(x, y)}, \quad \dot{y} = \frac{\mu}{\Phi(x, y)}, \quad (2.3)$$

where $\lambda, \mu = \text{const}$, $|\lambda| + |\mu| \neq 0$ and Φ is a smooth positive function 2π -periodical with respect to x and y .

Let us mention the main points of the proof. Since the vector field f is tangent to E_c , differential equation (2.1) is bounded on E_c . This equation on E_c has an integral invariant

$$\int \frac{M d\sigma}{V_{n-2}},$$

where $d\sigma$ is the element of area of E_c considered as a surface embedded into \mathbb{R}^n , V_{n-2} is the $(n-2)$ -dimensional volume of the parallelepiped in \mathbb{R}^n , the gradients of F_1, \dots, F_{n-2} being its sides. Now the integrability by quadratures on E_c follows from Euler's remark. The first conclusion of theorem 1 (which was firstly mentioned by Jacobi) is proved by this reasoning. The second conclusion is the well-known topological fact: any connected, compact, orientable, two-dimensional manifold that admits a tangent field without singular points is diffeomorphic to a two-dimensional torus. The third conclusion is, in fact, the Kolmogorov theorem on reduction of differential equations on a torus with a smooth invariant measure [9].

Equations (2.3) have invariant measure

$$\iint |\Phi(x, y)| dx \wedge dy.$$

By averaging the right-hand sides of (2.3) with respect to this measure we get the differential equations

$$\dot{u} = \frac{\lambda}{\nu}, \quad \dot{v} = \frac{\mu}{\nu}; \quad \nu = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \Phi dx dy. \quad (2.4)$$

Proposition 1. *Let $\Phi: T^2 \rightarrow \mathbb{R}$ be a smooth (analytical) function. Then for almost all pairs $(\lambda, \mu) \in \mathbb{R}$ there exists a smooth (analytical) change of angle variables $x, y \rightarrow u, v$ that reduces (2.3) to (2.4).*

The proof is presented in [9,10]. Note that if (2.3) can(not) be reduced to (2.4) for a pair (λ, μ) then the same is true for all pairs $(\kappa\lambda, \kappa\mu)$, $\kappa \neq 0$. So, the property of reducibility depends on arithmetical properties of $\frac{\lambda}{\mu}$ that is referred to as the rotation number of the tangent vector field on $T^2 = \{x, y \bmod 2\pi\}$.

Proposition 2. *Let $\Phi(x, y) = \sum \varphi_{m,n} \exp i(mx + ny)$, $\varphi_{m,n} = \overline{\varphi_{-m,-n}}$. If (2.3) may be reduced to (2.4) by a differentiable change of angle variables $u = u(x, y)$, $v = v(x, y)$ then*

$$\sum_{|m|+|n| \neq 0} \left| \frac{\varphi_{m,n}}{m\lambda + n\mu} \right|^2 < \infty. \quad (2.5)$$

If the ratio $\frac{\lambda}{\mu}$ is rational then the torus T^2 is stratified into a family of closed trajectories. In this case the reducibility condition is equivalent to the equality of periods of rotation for different closed trajectories.

In the general case (the Fourier decomposition of Φ contains harmonics) the points $(\lambda, \mu) \in \mathbb{R}^n$ with rationally independent (λ, μ) , for which series (2.5) diverges, are everywhere dense in \mathbb{R}^n . The questions of reducibility of (2.3) are discussed in [9]. The general properties of solutions of (2.3) see in [10].

3. S. A. Chaplygin's problem

Let us consider as an example the problem of rolling of a balanced, dynamically non-symmetric ball on a horizontal rough plane (see [5]). The motion of the ball is described by the following system of

equations in $\mathbb{R}^6 = \mathbb{R}^3\{\omega\} \times \mathbb{R}^3\{\gamma\}$:

$$\begin{aligned} \dot{k} + \omega \times k &= 0, \quad \dot{\gamma} + \omega \times \gamma = 0; \\ k &= I\omega + ma^2\gamma \times (\omega \times \gamma). \end{aligned} \quad (3.1)$$

Let ω be the vector of the angular rotation velocity of the ball, γ the unit vertical vector, I the tensor of inertia of the ball with respect to its center, m the mass of the ball, and a its radius. These equations have the invariant measure with density

$$M = \frac{1}{\sqrt{(ma^2)^{-1} - \langle \gamma, (I + ma^2 E)^{-1} \gamma \rangle}}, \quad E = \|\delta_{ij}\|. \quad (3.2)$$

Taking into account the existence of four independent integrals $F_1 = \langle k, \omega \rangle$, $F_2 = \langle k, \gamma \rangle$, $F_3 = \langle \gamma, \gamma \rangle$, $F_4 = \langle k, k \rangle$, we see that (3.1) is integrated by quadratures. Note that system of equations (3.1) has no equilibriums on the non-critical level sets E_c . Indeed, if $\gamma \dot{=} 0$, then ω and γ are linearly dependent. This fact implies the linear dependence of dF_1 and dF_2 . The simplest case of integration by quadratures of equations (3.1) is the case of zero value of the constant in the “area” integral F_2 . In elliptic coordinates ξ, η on the Poisson sphere $\langle \gamma, \gamma \rangle = 1$ the equations of motion on the level E_c are reduced to the following form

$$\begin{aligned} \dot{\xi} &= \frac{\sqrt{P_5(\xi)}}{\xi(\xi^{-1} - \eta^{-1})\Phi(\xi, \eta)}; \quad \dot{\eta} = \frac{\sqrt{P_5(\eta)}}{\eta(\xi^{-1} - \eta^{-1})\Phi(\xi, \eta)}; \\ \Phi &= \sqrt{(a - \xi)(a - \eta)}. \end{aligned}$$

Coefficients of the polynomial P_5 of the fifth order and the constant a depend on parameters of the problem and on the constants of the first integrals (see [5] for details). The variables ξ, η ranges over different closed intervals $a_1 \leq \xi \leq a_2$, $b_1 \leq \eta \leq b_2$ where P_5 is nonnegative. The uniformizing substitution

$$\begin{aligned} x &= \lambda \int_{a_1}^{\xi} \frac{z dz}{\sqrt{P_5(z)}}, \quad \lambda^{-1} = \frac{1}{\pi} \int_{a_1}^{a_2} \frac{z dz}{\sqrt{P_5(z)}}, \\ y &= \mu \int_{b_1}^{\eta} \frac{z dz}{\sqrt{P_5(z)}}, \quad \mu^{-1} = \frac{1}{\pi} \int_{b_1}^{b_2} \frac{z dz}{\sqrt{P_5(z)}}, \end{aligned} \quad (3.3)$$

introduces the angle variables $x, y \bmod 2\pi$ on E_c , and the equations of motion take form (2.3)

$$\begin{aligned} \dot{x} &= \frac{\lambda}{\Phi(x, y)}, \quad \dot{y} = \frac{\mu}{\Phi(x, y)}, \\ \Phi &= (\xi^{-1}(x) - \eta^{-1}(y)) \sqrt{(a - \xi(x))(a - \eta(y))}. \end{aligned} \quad (3.4)$$

Here $\xi(x)$ and $\eta(y)$ are 2π -periodic functions of x and y arising as the inversions of Abelian integrals (3.3).

These equations imply

Proposition 3. *The rotation number of a tangent vector field on two-dimensional invariant tori in Chaplygin’s problem is equal to the ratio of real periods of the Abelian integral*

$$\int \frac{z dz}{\sqrt{P_5(z)}}.$$

REMARK. This assertion is true for integrable problems of dynamics of a heavy body with a fixed point defined by the system of Euler-Poisson equations (see [10]). Since the Euler-Poisson equations are Hamiltonian, by the Liouville theorem, in integrable cases they always can be reduced to form (2.4) on two-dimensional invariant tori. It seems that equations (3.4) have no such property; inequality (2.5) is not fulfilled on all invariant non-resonant tori.

Let us make a change of time $t \rightarrow \tau$ by the formula

$$dt = \sqrt{(a - \xi)(a - \eta)} d\tau. \quad (3.5)$$

Equations (3.4) preserve their form but the variables x, y in function Φ are separated

$$\Phi = \xi^{-1}(x) - \eta^{-1}(y).$$

Proposition 4. Suppose that μ and λ in (2.3) are nonzero and

$$\Phi = \Phi'(x) + \Phi''(y).$$

Then equations (2.3) can be reduced to form (2.4) by an invertible change of angle variables on T^2 .

The proof is presented in [10]. Note that if $\Phi = \Phi'(x) + \Phi''(y)$, then series (2.5)

$$\sum_{n \neq 0} \left| \frac{\varphi'_n}{n\lambda} \right|^2 + \left| \frac{\varphi''_n}{n\mu} \right|^2$$

converges for all $\lambda, \mu \neq 0$.

So, taking into account change of time (3.5) one may reduce (3.4) to the form

$$\frac{du}{d\tau} = U, \quad \frac{dv}{d\tau} = V, \quad (3.6)$$

where U and V depend on the constants of the first integrals only, and $U, V \neq 0$. This result can cause the temptation to use Chaplygin's reducing multiplier theorem: if we can reduce (3.1) using change of time (3.5) to the Euler-Lagrange equations of some variational problem (they are written, as well as the classical Euler-Poisson equations, in redundant variables) then according to the Liouville theorem, the equations of motion on the two-dimensional invariant tori in some angle variables $u, v \bmod 2\pi$ have form (3.6). It is possible to show that this method does not lead to the goal. In conclusion, note that S. A. Chaplygin himself never considered the problem of the ball's rolling in connection with the reducing multiplier theory.

4. A generalization of S.A.Chaplygin's problem

We are going to show that the problem of rolling of a balanced, dynamically non-symmetric ball on a rough plane is still integrable (in the sense of section 1), if the particles of the ball are attracted by this plane proportionally to the distance. Since the center of mass of the ball coincides with its geometrical center, we can calculate the potential by the formula

$$V(\gamma) = \frac{\varepsilon}{2} \int \langle \mathbf{r}, \gamma \rangle^2 dm = \frac{\varepsilon}{2} \langle \mathbf{I}\gamma, \gamma \rangle, \quad (4.1)$$

where γ is the unit vertical vector, \mathbf{r} is the radius vector of particles of the ball, \mathbf{I} is the tensor of inertia of the ball with respect to its center. The attraction forces generate the rotational moment

$$-\int \mathbf{r} \times (\varepsilon \langle \mathbf{r}, \gamma \rangle \gamma) dm = -\varepsilon \int \langle \mathbf{r}, \gamma \rangle (\mathbf{r} \times \gamma) dm = \gamma \times V'_\gamma = \varepsilon \gamma \times \mathbf{I}\gamma.$$

In order to obtain the moment of forces with respect to the contact point, it is necessary to add the moment of the combined force

$$\varepsilon \int \langle \mathbf{r}, \gamma \rangle \gamma \, dm = \varepsilon \left\langle \int \mathbf{r} \, dm, \gamma \right\rangle \gamma,$$

which is equal to zero, since the center of mass of the ball coincides with its geometrical center.

According to the theorem about the change of kinetic moment behavior with respect to the contact point (see [5], [6]), the equations of rolling of the ball can be presented in the following form

$$\dot{\mathbf{k}} + \boldsymbol{\omega} \times \mathbf{k} = \varepsilon \gamma \times \mathbf{I} \gamma, \quad \dot{\gamma} + \boldsymbol{\omega} \times \gamma = 0. \quad (4.2)$$

Theorem 2. *Differential equations (4.2) are integrable by quadratures.*

Indeed, they have four independent integrals

$$\begin{aligned} F_1 &= \langle \mathbf{k}, \boldsymbol{\omega} \rangle + \varepsilon \langle \mathbf{I} \gamma, \gamma \rangle, & F_2 &= \langle \mathbf{k}, \gamma \rangle, & F_3 &= \langle \gamma, \gamma \rangle = 1, \\ F_4 &= \langle \mathbf{k}, \mathbf{k} \rangle - \langle \mathbf{A} \gamma, \gamma \rangle, \end{aligned}$$

where elements A_i of a diagonal matrix A are expressed through the principal moments of inertia I_i by the formulae

$$A_1 = \varepsilon(I_2 + ma^2)(I_3 + ma^2), \dots$$

Since equations (4.2) have the invariant measure with density (3.2), they are integrable by theorem 1. It would be interesting to integrate this equation explicitly and test if proposition 3 remains true for equations (4.2).

Note that the problem of rotation of a body about a fixed point in an axisymmetric force field with potential (4.1) is also integrable ([1]). In addition to the classical integrals F_1, F_2, F_3 , there is the integral F_4 , where one must put $\mathbf{a} = 0$. This integral was found independently by Clebsch in the problem on motion of a body in an ideal fluid and by Tisserand, who investigated rotational motion of heavenly bodies.

5. G. K. Suslov's problem and its generalization

Following G. K. Suslov ([11], ch. 53), we consider the problem of rotation about a fixed point of a body with the nonintegrable constraint $\langle \mathbf{a}, \boldsymbol{\omega} \rangle = 0$, where \mathbf{a} is a vector that is constant in the moving frame of reference. Suppose that the body rotates in an axisymmetric force field with the potential $V(\gamma)$. Following the method of Lagrange multipliers, we write down the equations of motion ([11], ch. 46):

$$\mathbf{I} \dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathbf{I} \boldsymbol{\omega} = \gamma \times V'_\gamma + \lambda \mathbf{a}, \quad \dot{\gamma} + \boldsymbol{\omega} \times \gamma = 0, \quad \langle \mathbf{a}, \boldsymbol{\omega} \rangle = 0. \quad (5.1)$$

Using the constraint equation $\langle \mathbf{a}, \boldsymbol{\omega} \rangle = 0$, the Lagrange factor can be expressed as the function of $\boldsymbol{\omega}$ and γ

$$\lambda = -\langle \mathbf{a}, \mathbf{I}^{-1}(\mathbf{I} \boldsymbol{\omega} \times \boldsymbol{\omega}) + \mathbf{I}^{-1}(\gamma \times V''_\gamma) \rangle / \langle \mathbf{a}, \mathbf{I}^{-1} \mathbf{a} \rangle.$$

Equations (5.1) always have three independent integrals:

$$F_1 = \langle \mathbf{I} \boldsymbol{\omega}, \boldsymbol{\omega} \rangle / 2 + V(\gamma), \quad F_2 = \langle \gamma, \gamma \rangle, \quad F_3 = \langle \mathbf{a}, \boldsymbol{\omega} \rangle.$$

For real motions, $F_2 = 1, F_3 = 0$. In this case, we can reduce the problem of integration of equations (5.1) to the problem of existence of an invariant measure (the existence isn't evident) and the fourth independent integral.

Proposition 5. *If \mathbf{a} is an eigenvector of operator \mathbf{I} , then the phase flow of system (5.1) preserves the "standard" measure in $\mathbb{R}^6 = \mathbb{R}^3\{\boldsymbol{\omega}\} \times \mathbb{R}^3\{\gamma\}$.*

To prove the proposition we have to verify the following fact: the divergence of the right-hand side of (5.1) is equal to zero as $\mathbf{I} \mathbf{a} = \mu \mathbf{a}$.

G.K. Suslov has considered a particular case of the problem, when the body is not under action of exterior forces: $V \equiv 0$. In this case the first equation of (5.1) is closed relatively to ω . We can show that it is integrable by quadratures (see [11], ch. 53). The analysis of these quadratures shows that if \mathbf{a} isn't an eigenvector of the inertia operator, then all trajectories $\omega(t)$ approach asymptotically as $t \rightarrow \pm\infty$ to some fixed straight line on the plane $\langle \mathbf{a}, \omega \rangle = 0$ (see Fig. 1). Consequently, the equation with respect to ω and complete system (5.1) have no invariant measure with continuous density. In this case theorem 1 isn't applicable, so, the question about the possibility to find the vector $\gamma(t)$ by quadratures remains open. But if $\mathbf{I}\mathbf{a} = \mu\mathbf{a}$ then equations (5.1) have the additional integral: the value of the kinetic moment is preserved

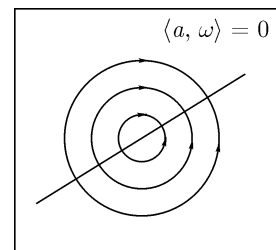


Fig. 1

$$F_4 = \langle \mathbf{I}\omega, \mathbf{I}\omega \rangle.$$

Equations (5.1) are integrable by theorem 1. However, this possibility may be easily realized directly. It seems that in the most general case, the existence of an invariant measure is connected with the hypothesis of proposition 5: $\mathbf{I}\mathbf{a} = \mu\mathbf{a}$. From now on, we suppose that this equality is fulfilled.

Now suppose that the body rotates in the homogeneous force field $V = \langle \mathbf{b}, \gamma \rangle$. If $\langle \mathbf{a}, \mathbf{b} \rangle = 0$, then equations (5.1) have the integral

$$F_4 = \langle \mathbf{I}\omega, \mathbf{b} \rangle$$

consequently, they are integrable by quadratures. This case was indicated by E.I. Kharlamova in her work [12]. We are going to consider an "opposite" case, when $\mathbf{b} = \varepsilon\mathbf{a}$, $\varepsilon \neq 0$. Without loss of generality we can assume that the vector \mathbf{a} has the components $(0, 0, 1)$. Taking into account the equation $\omega_3 = 0$, we obtain that two first equations (5.1) have the following form

$$I_1\dot{\omega}_1 = \varepsilon\gamma_2, \quad I_2\dot{\omega}_2 = -\varepsilon\gamma_1; \quad \omega = (\omega_1, \omega_2, \omega_3).$$

Therefore $I_1\ddot{\omega}_1 = \varepsilon\dot{\gamma}_2$, $I_2\ddot{\omega}_2 = -\varepsilon\dot{\gamma}_1$.

Using the Poisson equations $\dot{\gamma}_1 = -\omega_2\gamma_3$, $\dot{\gamma}_3 = \omega_1\gamma_3$ we get

$$I_1\ddot{\omega}_1 = \varepsilon\gamma_3\omega_1, \quad I_2\ddot{\omega}_2 = \varepsilon\gamma_3\omega_2. \quad (5.2)$$

The energy integral

$$(I_1\omega_1^2 + I_2\omega_2^2)/2 + \varepsilon\gamma_3 = h$$

makes it possible to express γ_3 through ω_1 and ω_2 . After that, equations (5.2) may be rewritten as the Lagrange equations

$$I_i^2\ddot{\omega}_i = \frac{\partial V}{\partial \omega_i} \Leftrightarrow \frac{d}{dt} \frac{\partial L}{\partial \dot{\omega}_i} = \frac{\partial L}{\partial \omega_i} \quad (i = 1, 2),$$

$$L = T_V, \quad T = \frac{I_1^2\dot{\omega}_1^2 + I_2^2\dot{\omega}_2^2}{2}, \quad V = \frac{1}{2} \left(h - \frac{I_1\omega_1^2 + I_2\omega_2^2}{2} \right)^2.$$

These equations have the energy integral $T + V$. For real motions its value is equal to $\varepsilon^2/2$. Let us emphasize that unlike the reducing multiplier theory our reduction of equation (5.1) to Lagrange (or Hamilton) equations doesn't require the change of time (cf. [11]).

The change $I_i\omega_i = k_i$ corresponding to the transition from the angular velocity to the kinetic moment reduces the considered problem of rotation of a body to the problem of motion of a material point in the potential force field

$$\ddot{k}_i = -\frac{\partial V}{\partial k_i} \quad (i = 1, 2), \quad V = \frac{1}{2} \left(h - \frac{k_1^2 I_1^{-1} + k_2^2 I_2^{-1}}{2} \right)^2. \quad (5.3)$$

At $I_1 = I_2$ we have the motion of point in a central field. This motion corresponds to the well-known integrable “Lagrange case” of the generalized Suslov problem. As well as in Lagrange’s classical problem of a heavy symmetric top, the equations of motion are integrable in this case in elliptical functions of time. If $I_1 \neq I_2$, then the equations apparently have no additional analytical integral independent of the energy integral. The following observation confirms this supposition. Put formally $I_1 = -I_2 = 1$. Then at $h = 0$ equations (5.3) practically coincide with the equations of the Young-Mills homogeneous two-component model, non-integrability of which is established in [14].

If the value of h is fixed, the point moves in the area defined by the inequality $V \leq \varepsilon^2/2$. For different h , these areas are shown in Fig 2. The trajectories of vibrational motions, when one of the components k_1 or k_2 becomes zero, are especially interesting. These motions are expressed through elliptical functions of time.

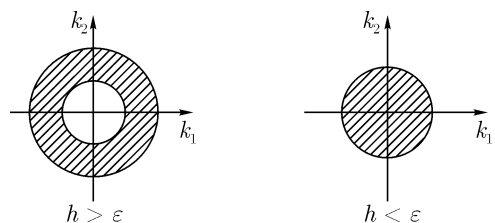


Fig. 2

One more case of integrability of (5.1) is given by

Theorem 3. Suppose **Ia** and the potential $V(\gamma)$ is defined by formula (4.1). Then equations (5.1) are integrable by quadratures.

Proof.

Let us show that equations (5.1) have the Clebsch–Tisseran integral

$$F_4 = \frac{1}{2} \langle I\omega, I\omega \rangle - \frac{1}{2} \langle A\gamma, \gamma \rangle, \quad A = \varepsilon I^{-1} \det I.$$

Indeed,

$$\begin{aligned} F_4 &= \langle I\omega, \gamma \times \varepsilon I\gamma \rangle + \lambda \langle a, I\omega \rangle + \langle A\gamma, \omega \times \gamma \rangle = \\ &= \langle \omega, I(\gamma \times \varepsilon I\gamma) \rangle + \lambda \mu \langle a, \omega \rangle + \langle \omega, \gamma \times A\gamma \rangle = \\ &= \langle \omega, I(\gamma \times \varepsilon I\gamma) + \gamma \times A\gamma \rangle = 0, \end{aligned}$$

since $I(\gamma \times I\gamma) = -(\gamma \times I^{-1}\gamma) \det I$. To complete the proof, we have to take into account the conclusion of proposition 5 and to use theorem 1.

Let us show, how one can explicitly integrate equation (5.1). For definiteness, let $a = (0, 0, 1)$ and $\varepsilon > 0$, $I_3 \geq I_1$, $I_3 \geq I_2$. Then (5.1) may be presented as the following closed system of four differential equations:

$$\begin{aligned} I_1 \dot{\omega}_1 &= \varepsilon(I_1 - I_2)\gamma_2\gamma_3, & I_2 \dot{\omega}_2 &= \varepsilon(I_3 - I_1)\gamma_1\gamma_3, \\ \dot{\gamma}_1 &= -\omega_2\gamma_3, & \dot{\gamma}_2 &= \omega_1\gamma_3, & \gamma_3^2 &= 1 - \gamma_1^2 - \gamma_2^2. \end{aligned}$$

Let us introduce the new time τ by formula $d\tau = \gamma_3 dt$ and denote the differentiating with respect to τ by prime. Then equations of motion take form of a linear system with constant coefficients

$$\begin{aligned} I_1 \omega'_1 &= \varepsilon(I_2 - I_3)\gamma_2, & \gamma'_2 &= \omega_1, \\ I_2 \omega'_2 &= \varepsilon(I_3 - I_1)\gamma_1, & \gamma'_1 &= -\omega_2. \end{aligned}$$

They can be presented in the equivalent form

$$\begin{aligned} \gamma''_1 + \lambda_1^2 \gamma_1 &= 0, & \gamma''_2 + \lambda_2^2 \gamma_2 &= 0, \\ \lambda_1^2 &= \varepsilon(I_3 - I_1)/I_2, & \lambda_2^2 &= (I_3 - I_2)/I_1. \end{aligned}$$

Put

$$\varphi_1 = -\arctg \frac{\lambda_1 \gamma_1}{\omega_2}, \quad \varphi_2 = \arctg \frac{\lambda_2 \gamma_2}{\omega_1}.$$

These variables are angle variables on two-dimensional invariant tori with

$$\varphi'_1 = \lambda_1, \quad \varphi'_2 = \lambda_2.$$

Consequently,

$$\dot{\varphi}_1 = \lambda_1/\Phi, \quad \dot{\varphi}_2 = \lambda_2/\Phi; \quad \Phi = (1 - c_1^2 \sin^2 \varphi_1 - c_2^2 \sin^2 \varphi_2)^{-1/2}.$$

The constants c_1 and c_2 ($c_1^2 + c_2^2 \leq 1$) can be expressed as functions of constant values of the energy integral and Clebsch–Tisseran integral. The remarkable property of this problem is the fact that the ratio of frequencies λ_1/λ_2 is independent of initial data and depends only on the constants of parameters of the problem. Consequently, if the number

$$\sqrt{\frac{(I_3 - I_1)I_1}{(I_3 - I_2)I_2}}$$

is rational, then all solutions are periodic; otherwise practically all trajectories aren't closed (except degenerated motions, when $\gamma_1 \equiv 0$ or $\gamma_2 \equiv 0$). Let $\varphi_s(0) = a_s$. Then

$$t = \int_0^\tau \frac{dx}{\sqrt{1 - c_1^2 \sin^2(\lambda_1 x + a_1) - c_2^2 \sin^2(\lambda_2 x + a_2)}}.$$

If $c_1 = 0$ (or $c_2 = 0$) then γ_1 and γ_2 (and consequently, $\omega_1, \omega_2, \gamma_3$) are elliptical functions of time. This conclusion is true in the case $\lambda_1 = \lambda_2$ (i. e., when $I_1 = I_2$ or $I_3 = I_1 + I_2$) for all c_1, c_2 . In the most general case the analytical character of the solutions is essentially more complex. In conclusion, note that series (2.5) diverges in this problem if the irrational ratio λ_1/λ_2 is approximated by rational numbers anomalously fast.

REMARK. Equations (5.1) are also integrable for potentials of the general form

$$V(\gamma) = \frac{1}{2}(c_{11}\gamma_1^2 + c_{22}\gamma_2^2 + c_{33}\gamma_3^2 + 2c_{12}\gamma_1\gamma_2).$$

Using the change of time $d\tau = \gamma_3 dt$, the equations of motion are reduced to the linear system

$$I_2\gamma_1'' = -\frac{\partial V}{\partial \gamma_1}, \quad I_1\gamma_2'' = -\frac{\partial V}{\partial \gamma_2}; \quad \tilde{V} = V|_{\gamma_3^2=1-\gamma_1^2-\gamma_2^2}.$$

In the general case, potential V does not have a simple physical interpretation.

6. The first integrals used as constraints

Let $L(\dot{x}, x, t)$ be a Lagrangian of a nonholonomic system that satisfies the following “regularity” condition: the quadratic form

$$\left\langle \frac{\partial^2 L}{\partial \dot{x}} \xi, \xi \right\rangle$$

is positively definite. In particular, $\det \|L''_{\dot{x}\dot{x}}\| \neq 0$. The constraints (not necessarily linear) are given by the equations

$$f_1(\dot{x}, x, t) = \dots = f_m(\dot{x}, x, t) = 0 \tag{6.1}$$

with independent co-vectors

$$\frac{\partial f_1}{\partial \dot{x}}, \dots, \frac{\partial f_m}{\partial \dot{x}}.$$

The equations of motion can be presented as the Lagrange equations with the multipliers

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = \sum_{s=1}^m \lambda_s \frac{\partial f_s}{\partial \dot{x}}, \quad f_1 = \dots = f_m = 0. \quad (6.2)$$

Proposition 6. *If the functions L, f_1, \dots, f_m satisfy the above conditions, then a unique solution of (6.2) corresponds to every initial state that is permissible by constraints (6.1).*

Indeed, under these suppositions the multipliers $\lambda_1, \dots, \lambda_m$ are smooth functions of \dot{x}, x, t by the explicit function theorem.

Now suppose equations (6.2) have a first integral $F(\dot{x}, x, t)$. We get the following

Proposition 7. *If the co-vectors $\frac{\partial F}{\partial \dot{x}}, \frac{\partial f_1}{\partial \dot{x}}, \dots, \frac{\partial f_m}{\partial \dot{x}}$ are independent then $x(t)$ is a solution of (6.2) with the constant value of the integral $F = c$ if and only if this function is a motion of mechanical system with the Lagrangian L and constraints $f_1 = \dots = f_m = f_{m+1} = c$, where $f_{m+1} = F - c$.*

The sufficiency is obvious for $x(t)$ satisfies (6.2) if we put $\lambda_{m+1} = 0$. On the contrary, let $x(t)$ be a solution of a system of form (6.2), where s ranges from 1 to $m+1$. Let $y(t)$ be the unique motion of system (6.2) with the initial data $y(0) = x(0), \dot{y}(0) = \dot{x}(0)$. Evidently, $F|_{y(t)} = c$. The function $y(t)$ as well as $x(t)$ satisfies the equations of motion of the extended system with $\lambda_{m+1} = 0$. To complete the proof, it remains to use the conclusion of proposition 6.

Let us discuss one of possible applications of proposition 7. Suppose f_i are linear with respect to velocities and constraints (6.1) are non-integrable. If equations of motion have the linear integral F , then equations

$$f_1 = \dots = f_m = f_{m+1} = 0 \quad (f_{m+1} = F - c)$$

may turn out to be completely integrable. In this case the study of motions that belong to the level set $F = c$ is reduced to the investigation of some holonomic system. We do not have to integrate here the constraint equations, since the variables can be considered as redundant one's, and the equations of motion may be written as Hamilton equations in redundant variables (see [11], [15]).

Let us consider as an example Suslov's problem in a homogeneous force field in the Kharlamova integrable case. The equations $\langle \mathbf{a}, \boldsymbol{\omega} \rangle = 0$ and $\langle \mathbf{I}\boldsymbol{\omega}, \mathbf{b} \rangle = 0$ form an integrable field of directions on the manifold of the rigid body positions (on the group $\text{SO}(3)$). Thus, Suslov's problem is reduced in this case to a system with one degree of freedom. Though the one-dimensional manifold of states of such system isn't closed in $\text{SO}(3)$ in the general case.

If the constraints are non-linear with respect to velocities, it is natural to use the energy integral

$$H(\dot{x}, x) = \frac{\partial L}{\partial \dot{x}} \dot{x} - L$$

as the first integral¹.

For example, let us consider the Appell–Hamel system with the Lagrangian

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + gz, \quad g = \text{const}$$

and the non-linear constraint

$$\dot{x}^2 + \dot{y}^2 = k^2 \dot{z}^2, \quad k = \text{const} \neq 0 \quad (6.3)$$

(see [16] and [17]). By the energy integral

$$\frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - gz = h$$

¹The equation of motion has the integral of energy, if the constraints are homogeneous with respect to velocities and the Lagrangian does not depend on time explicitly.

and (6.3) we get the equation of the “integrable” constraint

$$\frac{\dot{z}(1+k^2)}{2} - gz = h. \quad (6.4)$$

Consequently, the coordinate z changes with the constant acceleration $g/(1+k^2)$. Excluding non-linear integrable constraint (6.4) (i. e., considering z as a known function of time) we get a more simple system with two degrees of freedom, the Lagrangian

$$\tilde{L} = \frac{1}{2}(\dot{x}^2 + \dot{y}^2)$$

and the constraint $\dot{x}^2 + \dot{y}^2 = f(t)$, where $f = k^2 \dot{z}^2$ is a known quadratic function of time. The further integration may be easily fulfilled.

7. Symmetries of nonholonomic systems

We suppose that the vector field $\mathbf{v}(x) \neq 0$ is a symmetry field of a nonholonomic system with Lagrangian $L(\dot{x}, x)$ and constraints

$$f_1(\dot{x}, x) = \dots = f_m(\dot{x}, x) = 0,$$

if the phase flow g_v^s of the differential equation

$$\frac{dx}{dt} = \mathbf{v}(x)$$

preserves L and f_1, \dots, f_m .

Proposition 8. *A phase flow of a symmetry field converts solutions of a nonholonomic system to solutions of the same system.*

Proof.

By the theorem on rectification of trajectories, the phase flow g_v^s in some local coordinates x_1, \dots, x_n is the following one-parameter group

$$x_1 \rightarrow x_1 + s; \quad x_2 \rightarrow x_2, \dots, x_n \rightarrow x_n.$$

With respect to these variables, L and f_i do not depend on x_1 , consequently, the equations of motion do not contain this variable, too. This fact implies proposition 8.

In the case of integrable constraints, the symmetry field corresponds to a linear with respect to velocities first integral of the equations of motion. It is not so in the case of nonholonomic systems.

Proposition 9. *If g_v^s preserves the Lagrangian and v is the field of possible velocities, i. e.*

$$\frac{\partial f_1}{\partial \dot{x}} \mathbf{v} = \dots = \frac{\partial f_m}{\partial \dot{x}} \mathbf{v} = 0,$$

then the equations of motion have the first integral

$$\frac{\partial L}{\partial \dot{x}} = \text{const.}$$

This assertion (“the Noether theorem”) is discussed in [6], for example.

Theorem 4. Suppose the equations of motion (6.2) have $n - m$ first integrals f_{m+1}, \dots, f_n . If
1) at points of the set $E_c = \{f_1 = \dots = f_m = 0, f_{m+1} = c_{m+1}, \dots, f_n = c_n\}$ the Jacobian

$$\frac{\partial(f_1, \dots, f_n)}{\partial \dot{x}_1, \dots, \dot{x}_n},$$

is nonzero,

2) there exist fields $\mathbf{v}_1, \dots, \mathbf{v}_{n-1}$ that are linearly independent at all points E_c and generate a solvable Lie algebra with respect to the commutation operation, while their phase flows $g_{v_i}^s$ preserve L and f_1, \dots, f_n ,

3) there are no vectors $\dot{x} = \sum \lambda_s \mathbf{v}_s(x)$, $\lambda_s \in \mathbb{R}$ among solutions of the system of equations

$$f_1 = \dots = f_m = 0, \quad f_{m+1} = c_{m+1}, \dots, \quad f_n = c_n, \quad (7.1)$$

then solutions of (6.2) that belong to E_c are found by quadratures.

REMARK. In some cases the existence of first integrals of nonholonomic systems can be established by the following observation. Let $F(\dot{x}, x)$ be the first integral of a “free” holonomic system with Lagrangian L . This function is an integral of a nonholonomic system with the same Lagrangian L and constraints $f_1 = \dots = f_m = 0$ in the case of

$$\left(\frac{\partial^2 L}{\partial \dot{x}^2}\right)^{-1} \frac{\partial f_s}{\partial \dot{x}} \cdot \frac{\partial F}{\partial \dot{x}} = 0, \quad 1 \leq s \leq m,$$

if $f_1 = \dots = f_m = 0$. This condition of invariance is fulfilled for the Clebsch-Tisseran integral in Suslov’s problem (theorem 3). Besides, it is fulfilled for the energy integral in the case of homogeneous constraints and for the Noether integral $\frac{\partial L}{\partial \dot{x}} \cdot \mathbf{v}$, if the field \mathbf{v} is the field of possible velocities (proposition 9).

Proof of theorem 4.

By the explicit function theorem we obtain from (7.1) that

$$\dot{x} = \mathbf{v}_c(x). \quad (7.2)$$

By conditions 2 and 3, the vectors $\mathbf{v}_c, \mathbf{v}_1, \dots, \mathbf{v}_{n-1}$ are linearly independent at all points x . The phase flows $g_{v_i}^s$ convert solutions of (7.2) to solutions of the same equation (proposition 8). To complete the proof, it remains to apply the well-known Lie theorem on integrability by quadratures of systems of differential equations (see, for example, [18]).

Let us consider as an illustrating example the problem of sliding of a balanced skate on horizontal ice. One can choose units of length, time and mass so that the Lagrangian would take the following form:

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2). \quad (7.3)$$

Here x, y are the coordinates of the point of contact, z is the angle of rotation of the skate. The constraint equation is

$$f = \dot{x} \sin z - \dot{y} \cos z = 0. \quad (7.4)$$

The equations of motion have two first integrals

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = h, \quad \dot{z} = c. \quad (7.5)$$

The second one is obtained by using proposition 9 with the help of the vector field $\mathbf{v}_3 = (0, 0, 1)$. By (7.4) and (7.5) we obtain the field $\mathbf{v}_{h,c} = (\sqrt{h - c^2} \cos z, \sqrt{h - c^2} \sin z, c)$. The fields $\mathbf{v}_1 = (1, 0, 0)$ and $\mathbf{v}_2 = (0, 1, 0)$ are commuting symmetry fields. If $c \neq 0$ then vectors $\mathbf{v}_{h,c}, \mathbf{v}_1$ and \mathbf{v}_2 are linearly independent, consequently, in this case we can apply theorem 4. Let us emphasize that \mathbf{v}_1 and \mathbf{v}_2 do not generate the conservation laws.

Theorem 4 impose strict restrictions on the nonholonomic system. These restrictions can be weakened if we replace condition 2 by the condition

2) for the fixed $c = (c_{m+1}, \dots, c_h)$ there exist $n - 1$ linearly independent fields $v_i(x, c)$ that generate a solvable Lie algebra and commute with $v_c(x)$.

Let us add to Lagrangian (7.3) the term $-x/2$. Thus, we have placed the skate onto an inclined plane. Equations (7.4)–(7.5) hold if we replace x by $h + x$. Then the field $v_{h,c}$ is equal to

$$(\sqrt{h - c^2 + x} \cos z, \sqrt{h - c^2 + x} \sin z, c).$$

If h and $c \neq 0$ are fixed, then the fields

$$v_1 = (2\sqrt{h - c^2 + x}, -(\cos z)/c, 0), \quad v_2 = (0, 1, 0)$$

and $v_{h,c}$ are independent, and all their commutators vanish. In the same way one can solve the problem of rolling of a homogeneous disk on a rough plane, the problem of rolling of a ball in a vertical pipe and a series of other problems of nonholonomic mechanics.

8. Existence of an invariant measure

The existence of an integral invariant with a positive density is interesting not only from the standpoint of integration of differential equations. It is interesting by itself, from the standpoint of possible applications, for example, in ergodic theory. We are going to consider the problem of existence of an invariant measure for systems of differential equations. We are especially interested in its applications to nonholonomic mechanics.

By the theorem on rectification of trajectories, in a sufficiently small neighborhood of an ordinary point there always exists an invariant measure with a smooth stationary density. Therefore, the problem of existence of an invariant measure is especially interesting near equilibria as well as in sufficiently big domains of the phase space, where trajectories have the property of returning. Let us consider the first possibility. Let the point $x = 0$ be an equilibrium of an analytical system of differential equations

$$\dot{x} = \Lambda x + \dots \quad (8.1)$$

We say that a set of (complex) eigenvalues $\lambda_1, \dots, \lambda_n$ of the matrix Λ is resonant, if $\sum m_i \lambda_i = 0$ for some natural m_i . Note that a weaker resonance condition: $\sum m_i \lambda_i = 0$ for some integer $m_i \geq 0$ and $\sum |m_i| \neq 0$ is usually used for investigation of system (8.1) (for example, in the theory of normal forms).

Proposition 10. *If a set $\lambda_1, \dots, \lambda_n$ is not resonant then in a small neighborhood of the point $x = 0$ equation (8.1) has no integral invariant with an analytical density.*

The non-resonance condition is fulfilled, for example, in the case of $\operatorname{Re} \lambda_i \geq 0$ (≤ 0) and $\sum \operatorname{Re} \lambda_i > 0$ (< 0).

Proof.

Let us expand the density $M(x)$ in a convergent series with respect to homogeneous forms:

$$M = M_s + M_{s+1} + \dots, \quad s \geq 0.$$

Evidently, M_s is the density of the integral invariant for the linear system $\dot{x} = \Lambda x$. One can assume that Λ as already reduced to the canonical Jordan form. Let us arrange the monomials of the form M_s in some lexicographical order:

$$M_s = \sum_{\substack{m_i \geq 0 \\ m_1 + \dots + m_n = s}} a_{m_1 \dots m_n} x_1^{m_1} \dots x_n^{m_n}.$$

It is evident that $\operatorname{div} M_s(\Lambda x)$ is some form of the same degree. By equating its coefficients to zero, we get a linear homogeneous system of equations with respect to $a_{m_1 \dots m_n}$. The determinant of this system is equal to the product

$$\prod_{\substack{m_i \geq 0 \\ m_1 + \dots + m_n = s}} [(m_1 + 1)\lambda_1 + \dots + (m_n + 1)\lambda_n].$$

This product is nonzero by supposition. Consequently, all $a_{m_1 \dots m_n} = 0$. ■

REMARK. If a more strict condition of absence of resonant ratios in the traditional sense is fulfilled then equation (8.1) has no first integrals analytical in a neighborhood of the point $x = 0$.

Let us consider as an example the problem on permanent rotations of a convex rigid body with an analytical convex bound on a horizontal absolutely rough plane (see [4]). The motion of such body is described by a system of six differential equations that have the integral of energy and the geometrical integral. In a particular case, when one of principal central axes of inertia of the body is orthogonal to its surface, we have a one-parameter family of stationary rotations about the vertical axis of inertia. Singular points of equations of motion correspond to the stationary motions. The characteristic equation has the following form:

$$\lambda^2(a_4\lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda^1 + a_0) = 0.$$

The dependence of the coefficients a_s on numerous parameters of the problem is rather complicated; practically, they are arbitrary. The existence of the double zero root is connected with the existence of two independent integrals, since the differentials of the energy integral and of the geometrical integral are independent in the general case at points that correspond to permanent rotations. Fixing the levels of the first integrals we get differential equations on four-dimensional manifolds that have no invariant measure with an analytical density in the general case. Consequently, the initial equations also have no invariant measure in a neighborhood of stationary motions.

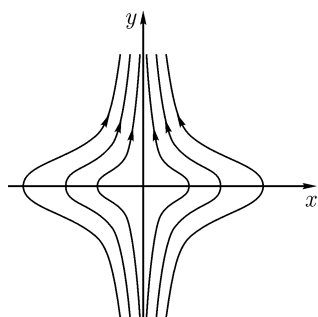


Fig. 3

Now let us consider the problem of existence of invariant measure for systems of differential equations that are similar to integrable systems that satisfy the conditions of theorem 1. It is natural to take the constants of the first integrals I_1, \dots, I_{n-2} as independent variables in a neighborhood of invariant tori of a non-disturbed integral system and to take angle variables $x, y \bmod 2\pi$ on the invariant tori. In these variables, the perturbed system has the following form:

$$\begin{aligned} \dot{I}_s &= \varepsilon f_s(I, x, y) + \dots, \quad s = 1, \dots, n-2, \\ \dot{x} &= \frac{\lambda(I)}{\Phi(I, x, y)} + \varepsilon X(I, x, y) + \dots, \quad \dot{y} = \frac{\mu(I)}{\Phi(I, x, y)} + \varepsilon Y(I, x, y) + \dots \end{aligned} \quad (8.2)$$

We assume that all functions in the right-hand sides of these differential equations are analytical in the direct product $D \times T^2$, where D is a domain in $\mathbb{R}^{n-2} = \{I_1, \dots, I_{n-2}\}$, $T^2 = \{x, y \bmod 2\pi\}$; ε is a small parameter. For system (8.2), it is natural to consider the problem of existence of an invariant measure, the density of which is analytical with respect to I, x, y , 2π -periodic with respect to x, y and analytically depends on ε :

$$M = M_0 + \varepsilon M_1 + \varepsilon^2 M_2 + \dots \quad (8.3)$$

The unperturbed problem has an invariant measure with the density M_0 . According to the well-known averaging principle, we average the right-hand sides of (8.2) with respect to the measure

$dm = \Phi dx \wedge dy$. As a result, we get the closed system of equations for changing of slow variables I in the domain D

$$\begin{aligned} \dot{I}_s &= \varepsilon F_s(I), \quad 1 \leq s \leq n-2, \\ F_s &= \frac{1}{\Lambda} \int_0^{2\pi} \int_0^{2\pi} f_s \Phi dx dy, \quad \Lambda = \int_0^{2\pi} \int_0^{2\pi} \Phi dx dy. \end{aligned} \quad (8.4)$$

Proposition 11. Suppose $m\lambda(I) + n\mu(i) \neq 0$ in domain D for all integer m, n that are not equal to zero simultaneously. If averaged system (8.4) has no invariant measure with analytical density, then initial system (8.2) also has no invariant measure with density (8.3).

System (8.4) is simpler than (8.2); the sufficient condition of nonexistence of an invariant measure for (8.4) is given by proposition 10.

Proof of proposition 11.

Coefficients M_0 and M_1 of (8.3) satisfy equations

$$\lambda \frac{\partial}{\partial x} \frac{M_0}{\Phi} + \mu \frac{\partial}{\partial y} \frac{M_0}{\Phi} = 0, \quad (8.5)$$

$$\sum_s \frac{\partial}{\partial I_s} (M_0 f_s) + \frac{\partial}{\partial x} M_0 X + \frac{\partial}{\partial y} M_0 Y = - \left(\lambda \frac{\partial}{\partial x} \frac{M_1}{\Phi} + \mu \frac{\partial}{\partial y} \frac{M_1}{\Phi} \right). \quad (8.6)$$

Since λ/μ is irrational for almost all $I \in D$, equation (8.5) implies $M_0 = \Gamma(I)\Phi$. Substituting this relation into (8.6) and averaging with respect to x, y we get the following equation:

$$\sum_s \frac{\partial}{\partial I_s} \Gamma F_s = 0. \quad (8.7)$$

Consequently, Γ is the density of the integral invariant of (8.4). It remains to show that $\Gamma \neq 0$. If it is not true then $M_0 = 0$. But in this case the function $M_1 + \varepsilon M_2 + \dots$ is the density of an invariant measure for (8.2) if $M_1 \equiv 0$, this operation may be repeated once more. The proposition is proved.

REMARK. One can show that (under conditions of proposition 11) if averaged system (8.4) has no analytical first integral in D then initial system (8.2) has no integral that can be expressed as a series $g_0 + \varepsilon g_1 + \dots$ with coefficients g_s analytical in $D \times T^2$.

Let us consider in more details the particular case, when $n = 3$. The index s may be omitted. Let $F(I) \neq 0$. If $F(I) = 0$ at some point of the interval D then (8.4) evidently has no invariant measure. Therefore, we assume that $F(I) \neq 0$ in D . Consider the following Fourier expansions:

$$\begin{aligned} \frac{X\Phi}{F} &= \sum X_{mn}(I) \exp i(mx + ny), \\ \frac{Y\Phi}{F} &= \sum Y_{mn}(I) \exp i(mx + ny), \\ \frac{f\Phi}{F} &= \sum f_{mn}(I) \exp i(mx + ny). \end{aligned}$$

The resonant set Δ is the set of points $I \in D$, such that

$$\sum_{|m|+|n| \neq 0} \left| \frac{a_{mn}}{m\lambda + n\mu} \right|^2 = \infty, \quad a_{mn} = \frac{df_{mn}}{dI} + i(mX_{mn} + nY_{mn}).$$

Proposition 12. Suppose

- 1) $\lambda(I)/\mu(I) \neq \text{const}$,
- 2) the intersection $\Delta \cap D$ is not empty.

Then (8.2) has no integral invariant with density (8.3).

Indeed, correlation (8.7) implies $\Gamma = c/F$, where $c = \text{const.}$ Let

$$\frac{M_1}{\Phi} = \Sigma b_{mn}(I) \exp i(mx + ny).$$

Equation (8.6) gives us the set of correlations

$$-(m\lambda + n\mu)b_{mn} = ca_{mn}.$$

Let $I \in \Delta$. Then the condition

$$\Sigma |b_{mn}|^2 < \infty$$

implies $c = 0$. ■

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